

Chapter 4 ORBITAL MECHANICS INTERLUDE

A Gravitational Force

Orbital mechanics is all about gravity, of course. You may be familiar with the relatively simple textbook case where the force due to gravity is given by the formula: $f = mg$, where m is the mass (generally in kilograms), and g is the acceleration due to gravity (approximately $9.8 \frac{m}{s^2}$). Unfortunately, this simple form, appropriate near the surface of the earth, will not do for orbital motion, and a more complex form must be used. We find that the correct formula for this '**central force problem**' is:

$$\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r} \quad (\text{Eqn. 4.1})$$

where $G = 6.67 \times 10^{-11} \text{ N} \frac{\text{m}^2}{\text{kg}^2}$ (Gravitational constant), m_1 , and m_2 are the masses involved (the earth and the satellite, generally), r is the separation between the center of the earth and the satellite, and the vector elements (and sign) indicate that the force is along a line joining the centers (of mass) of the two bodies. As always, the force F is in Newtons, the masses again are in kilograms. At the surface of the earth, this equation becomes the familiar form:

$$F = g_o m \quad (\text{Eqn. 4.2})$$

where $g_o = G \frac{m_{\text{Earth}}}{R_{\text{Earth}}^2} = 9.8 \frac{m}{s^2}$ is the acceleration due to gravity at the earth's surface. Note that this can lead to a relatively convenient form of Eqn. 4.1:

$$F = g_o m \left(\frac{R_{\text{Earth}}}{r} \right)^2 \quad (\text{Eqn. 4.3})$$

where $R_{\text{earth}} = 6380 \text{ km}$, and we have used $m_{\text{earth}} = 5.9736 \times 10^{24} \text{ kg}$.

[Rees (1990) notes that although the mass of the earth and G are not known to high accuracy, the product is: $GM_{\text{Earth}} = (3.98600434 \pm 2 \times 10^{-8}) \times 10^{14} \text{ m}^3 \text{s}^{-2}$. Note that the $\pm 2 \times 10^{-8}$ is the error in the last digit of the expression – we have 9 significant digits.

B Circular Motion

The force due to gravity results in a variety of possible solutions of the equations of motion. The simplest of these are the circular orbits, which objects like the moon approximate.

1 Equations of motion

The velocity of an object moving in a circular orbit is described by means of an **angular velocity** which determines the relationship between the radius of the circular motion, and the linear velocity.

$v = \omega r$, where

v = velocity in meters/second,

r = distance from center of motion, ω =angular velocity (radians/second).

(If you are used to 'regular' frequency, f , then $\omega = 2\pi f$)

Frequency is related to the period, τ , by the relation:

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega} \quad (\text{Eqn. 4.4})$$

a Example: Car going around in a circle of 200 m radius, at 36 km/hour. What is ω ?

$$\omega = \frac{v}{r} = \frac{(36 \times 10^3 \text{ m} / 3600 \text{ s})}{200 \text{ m}} = 0.05 \text{ radians/s}$$

b Example: a satellite is going around the earth once every 90 minutes. What is ω ?

Period (τ) = 90*60=5400 seconds.

$$f = \frac{1}{\tau} = \frac{1}{5400} = 1.85 \times 10^{-4} \text{ s}^{-1}; \quad \omega = 2\pi f = 1.16 \times 10^{-3} \text{ radians/s}$$

2 Centripetal Force:

Newton said that if a mass is going to move in a trajectory other than a straight line, a force must be applied. In particular, circular motion requires the application of a force termed a 'centripetal force', or

$$F_{\text{centripetal}} = m \frac{v^2}{r} = m \omega^2 r \quad (\text{Eqn. 4.5})$$

C Satellite Motion:

A consequence of the centripetal force applied to a satellite by gravity, for an object in circular motion, is the following balance:

$$F_{\text{centripetal}} = m \frac{v^2}{r} = F_{\text{gravity}} = g_o m \left(\frac{R_{\text{Earth}}}{r} \right)^2 \quad (\text{Eqn. 4.6})$$

Note that the mass cancels out - the orbital motion does not depend on the mass of the satellite.

$$\begin{aligned} \frac{v^2}{r} &= g_o \left(\frac{R_{\text{Earth}}}{r} \right)^2 \Rightarrow v^2 = \frac{g_o R_{\text{Earth}}^2}{r} \Rightarrow \\ v &= \sqrt{\frac{g_o}{r}} R_{\text{Earth}} \quad (\text{Eqn. 4.7}) \end{aligned}$$

There is a direct, inverse, relationship between the radius and velocity of a satellite in a circular orbit around the earth. This simple derivation is sufficient to introduce some of the most basic concepts of orbital motion - in particular Kepler's laws.

1 Illustration: Geosynchronous Orbit

What is the radius of the orbit of a geo-synchronous satellite - that is, a satellite with an

orbital period of 24 hours. First: $w = \frac{2p}{24 \text{ hours}} = \frac{2p}{86400 \text{ s}}$

$$v = \sqrt{\frac{g_o}{r}} R_{\text{Earth}} \Rightarrow w = \frac{v}{r} = \sqrt{\frac{g_o}{r^3}} R_{\text{Earth}} \Rightarrow \frac{w^2}{R_{\text{Earth}}^2} = \frac{g_o}{r^3}, \text{ or}$$

$$\frac{r^3}{R_{\text{Earth}}^3} = \frac{g_o}{w^2} \frac{1}{R_{\text{Earth}}^1} \Rightarrow$$

$$\frac{r}{R_{\text{Earth}}} = \left(\frac{g_o}{R_{\text{Earth}} w^2} \right)^{\frac{1}{3}} = \left(\frac{9.8}{6.38 \times 10^6 \frac{(2p)^2}{86400^2}} \right)^{\frac{1}{3}} = \left(290.45 \right)^{\frac{1}{3}} = 6.62$$

so we see that geo-synchronous orbit is 6.6 earth radii (geocentric)

What is the velocity of the satellite ?

2 Kepler's Laws

There are three "Kepler's Laws"

- the orbits are ellipses, with one focal point at the center of the earth,
- equal areas are swept out in equal times, and
- the square of the orbital period is proportional to the cube of the semi-major axis.

a Elliptical Orbits (circular is a special case)

An ellipse is characterized by the semi-major and semi-minor axes (a , b), or alternatively, the semi-major axis and the eccentricity (a , ϵ or e). In our case, the important point for orbital motion is the focus, which for satellite motion around the earth is the center of the earth.

Some useful formulas:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$e = \frac{\sqrt{a^2 - b^2}}{a} \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

Distance from center to focus is
 $c = e a = \sqrt{a^2 - b^2}$

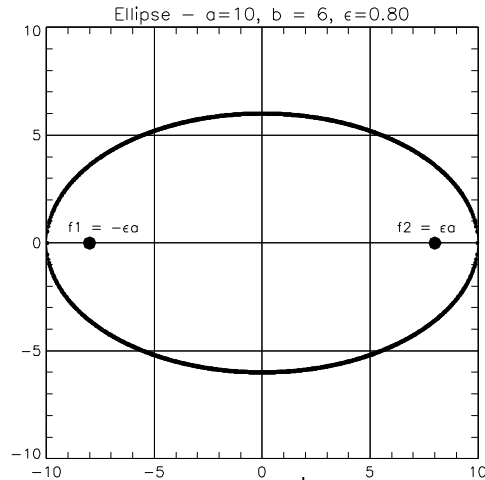


Figure 4-1 $r = \frac{a(1 - e^2)}{1 + e \cos \theta}$

b Equal areas are swept out in equal times

This law is a consequence of the conservation of angular momentum;

$$\vec{L} = m\vec{v} \times \vec{r}; \quad |\vec{L}| = mvr \sin \theta \quad \text{is a constant.}$$

Hence, at each point along the orbit, the product of the velocity perpendicular to the radial vector v_{\perp} , and the radius is a constant. Note that at perigee and apogee, the radial velocity is zero (by definition), and if one checks the values to the right, one can see that:

$$2.709 \cdot 6.192 = 13.277 \cdot 1.263$$

A consequence of this law is that the satellite spends the great majority of its time at apogee.

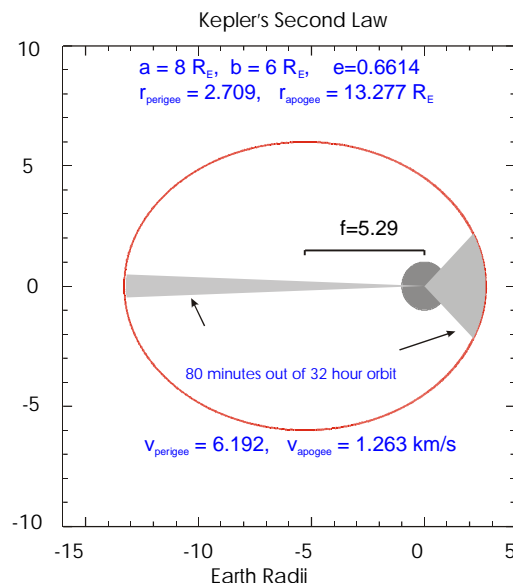


Figure 4-2: Earth is at one focus, $x=5.29$; the x range is -13.29 to $2.71 R_E$.

Note that the instantaneous velocity is given by:

$$v = \sqrt{GM \left(\frac{2}{r} - \frac{1}{a} \right)} \quad \text{where } r \text{ is the instantaneous radius, and } a \text{ is the semi-major axis.}$$

c Orbital Period: $t^2 \propto r^3$

As indicated above in the derivation of the period for a geosynchronous orbit,

$$v = \sqrt{\frac{g_o}{r}} R_{Earth} \Rightarrow w = \frac{v}{r} = \sqrt{\frac{g_o}{r^3}} R_{Earth} = \frac{2p}{t}$$

$$\text{or } t = \frac{2p}{R_{Earth}} \sqrt{\frac{r^3}{g_o}} \Rightarrow t^2 = \frac{4p^2}{g_o R_{Earth}^2} r^3 = \frac{4p^2}{M_{earth} G} r^3 \quad (\text{Eqn. 4.8})$$

This result is obtained quickly here for a circular orbit, but is more generally true. Replace the radius of the circle with the semi-major axis, and you get the value of the orbital period.

D Orbital Elements

1 Semi-Major Axis – a

The size of the orbit is determined by this parameter, as illustrated in the above 2 figures. A is half of the longest axis of the ellipse. A related measure of size is the distance to the focus, c. ($c = 8$ and 5.29 in the two ellipses shown above.)

2 Eccentricity – e or ϵ

This parameter determines the shape of the orbit, $e = c/a$. For a circle, $\epsilon=0$, for a straight line, $\epsilon=1$. (The latter would be a ballistic missile – straight up and straight down.)

3 Inclination Angle - I

This is the angle between the orbit plane and the equatorial plane of the earth. In the idealized case of a spherical earth, a geostationary satellite, at the earth's equator, would have an inclination of 0° . Reality differs slightly from this for modern geosynchronous satellites.

A polar orbiting satellite will have an inclination of 90° (more or less). The non-spherical earth exerts a torque on the satellite orbit plane. For an inclination of less than 90° , the orbit will precess west, for an inclination greater than 90° , the orbit will precess east. For the right choice, relative to the altitude, a polar orbiting satellite with the right inclination (greater than 90°) will be sun-synchronous- nominally 97° - 98° for LEO satellites.

The remaining parameters determine the relative phase of the orbit.

4 Right Ascension of the ascending node - Ω

The right ascension of the ascending node is the angle between the plane of the satellite orbit and the line connecting the earth and the sun when the earth is located at the vernal equinox (first day of spring). (Alternately described as being measured from Aries.)

Alternative descriptions: the latitude at which the satellite crosses the terrestrial equator (longitude of the ascending node), or the point on the celestial equator (right ascension of the ascending node or celestial longitude of the ascending node – measured from Aries.)

5 Closest point of approach (argument of perigee or ω)

This is the latitude for perigee, measured from the ascending node in the orbital plane in the direction of the satellite's motion. $\omega=0^\circ$ corresponds to perigee occurring over the equator, 90° puts perigee over the northern pole. Again, because of the non-spherical earth, the argument of perigee precesses.

Inclination $< 63.4^\circ$	ω precesses opposite satellite motion
Inclination $= 63.4^\circ$	ω does not precess (Molniya orbit)
Inclination $> 63.4^\circ$	ω precesses in the same direction as the satellite motion

E A few standard Orbits

There are a set of reasonably standard orbits used in the satellite industry, most of which have some use in the remote sensing community. In range of altitude, they range from low-earth-orbit (LEO) at altitudes of a few hundred miles, up to geosynchronous orbit, at an altitude of some 20,000 miles.

1 Low-earth orbit (LEO)

LEO is the domain of a large fraction of remote sensing satellites of various kinds: weather, earth resources, and reconnaissance. These satellites are typically in a sun-synchronous orbit. This means that they cross the equator at the same local time during each orbit, in order to maintain a constant solar illumination angle for their observations.

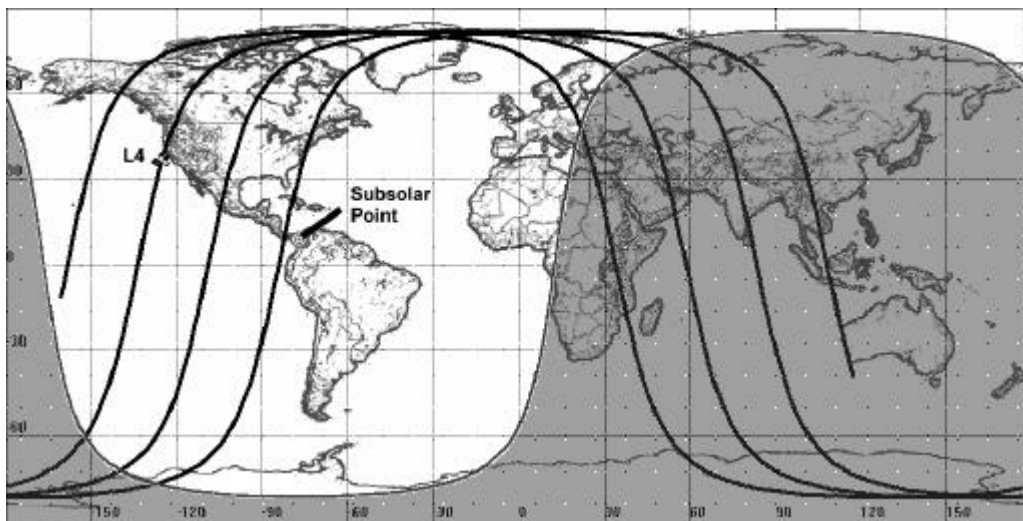


Figure 4.3 Ground track for 3 orbits by a LEO satellite, Landsat 4, crossing the equator during each orbit at 0940 local time. (Note the solar sub-point just above the coast of South America, corresponding to the time of the satellite crossing the equator). In the 98 minute orbit, the earth has rotated 24.5 degrees.

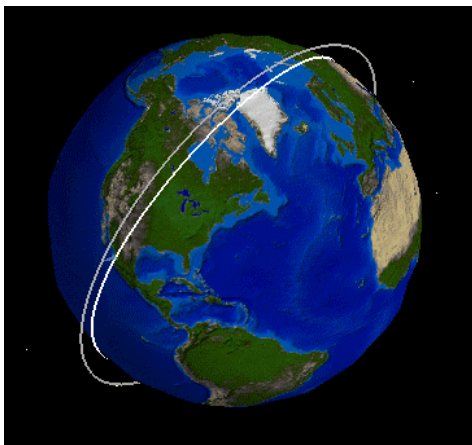


Figure 4.4a

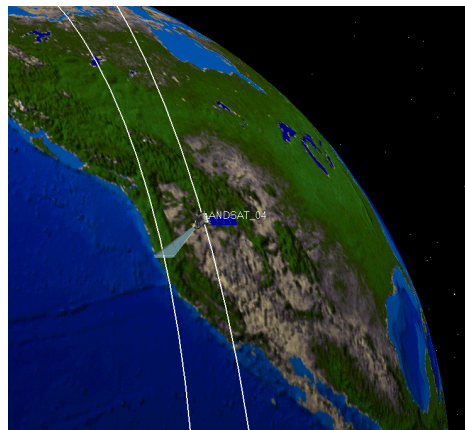


Figure 4.4b The sensor on Landsat 4 sweeps along the orbit, aimed at the sub-satellite point (in the nadir direction) Over 15 days, the satellite will have observed the entire earth.

2 Medium Earth Orbit (MEO)

MEO is the domain of the Global Positioning Satellites (GPS). Though not directly used for remote sensing, they are increasingly important for mapping, and so do influence the interpretation of remotely sensed imagery. These satellites are in $4.15 R_E$ circular (26378 km geocentric) orbits, with 12 hour periods.

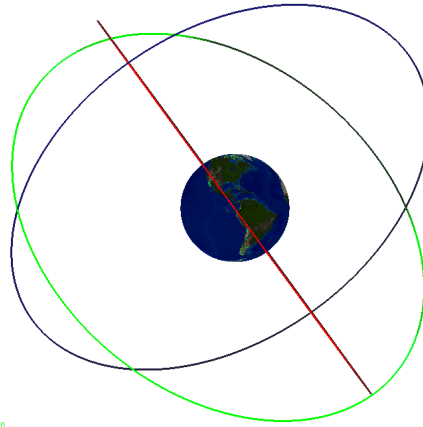


Figure 4.5

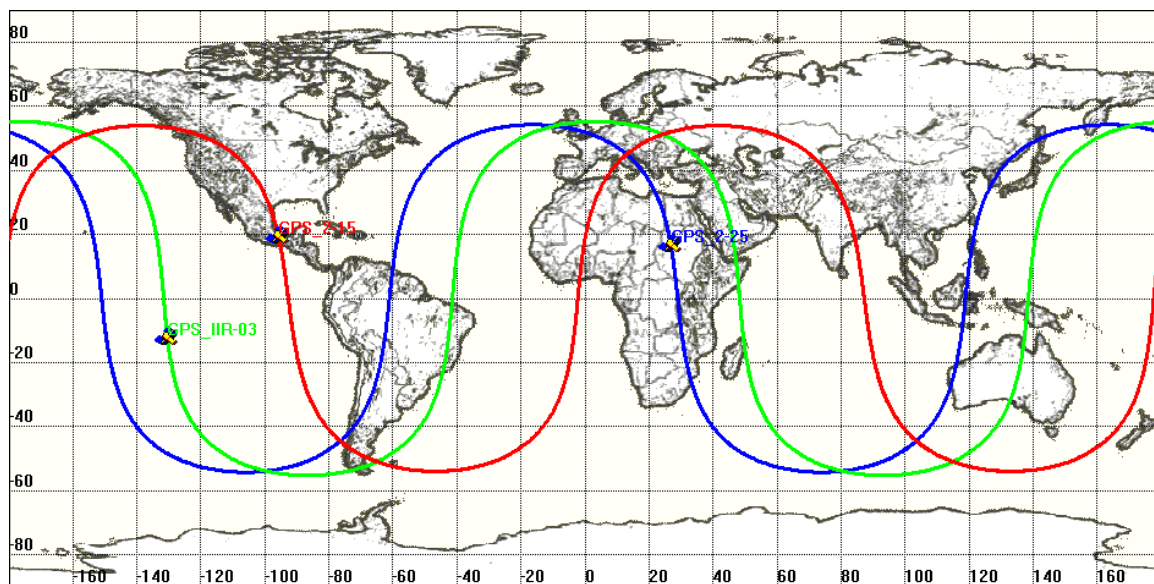


Figure 4.6 Orbit tracks for 3 GPS satellites

3 Molniya (HEO)

The Molniya, or HEO orbits are useful for satellites that need to 'dwell' at high latitudes for an extended period. The careful match of inclination and eccentricity allows a balance of forces that keeps the orbit plane from precessing, that is, the latitude at which apogee occurs does not vary. This is the standard orbit for Russian communications satellites.

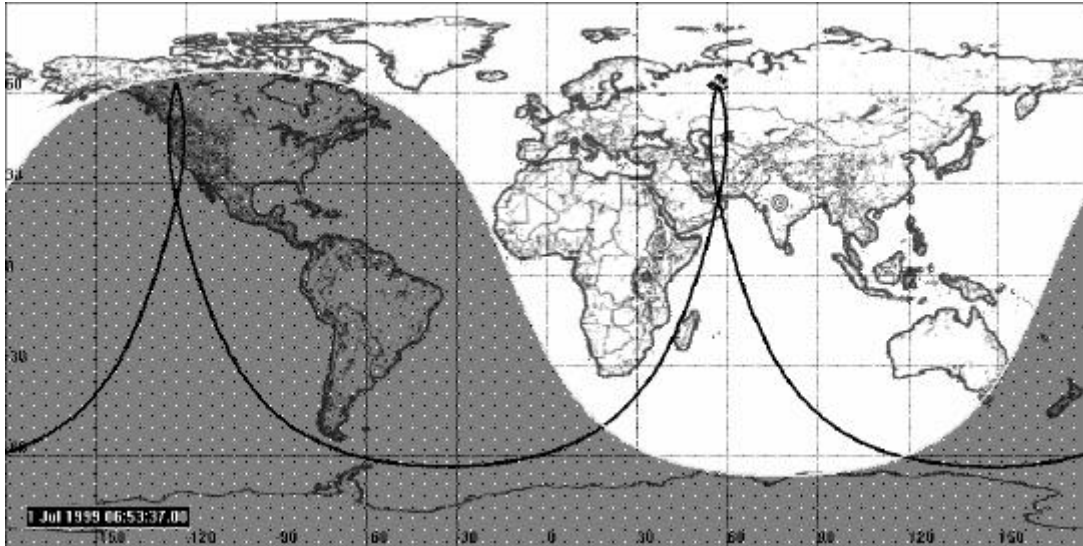


Figure 4.7 Molniya orbit, The subsolar point is centered on India.

Semi-major axis: a	26553.375 km	apogee radius	46228.612 km
Eccentricity: e	0.74097	perigee radius	6878.137 km
Inclination: i	63.40°	perigee altitude	500.000 km
argument of perigee	270.00°	RAAN	335.58°
Longitude of ascending node	230.043°	mean motion (revs per day)	2.00642615
period	43061.64 sec	Epoch	1 July 1999, 00:00:00

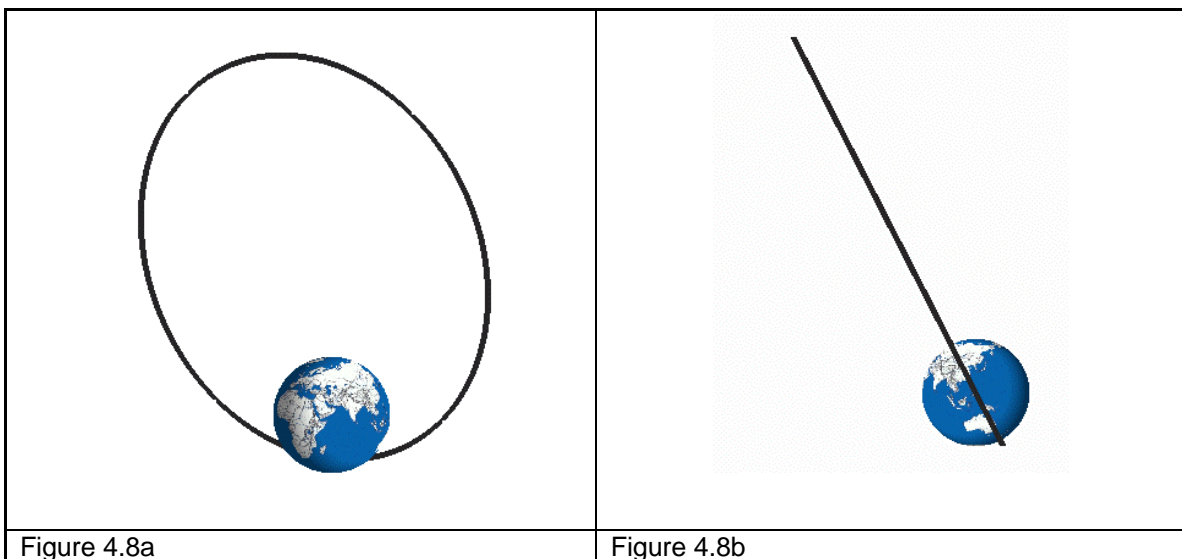


Figure 4.8a

Figure 4.8b

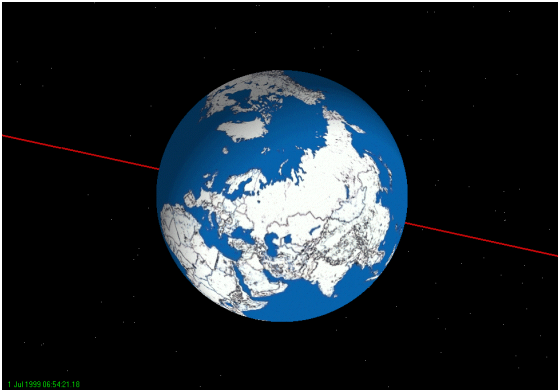


Figure 4.9a The view from Molniya orbit, corresponding to the location at apogee illustrated above. (06:54 UT)

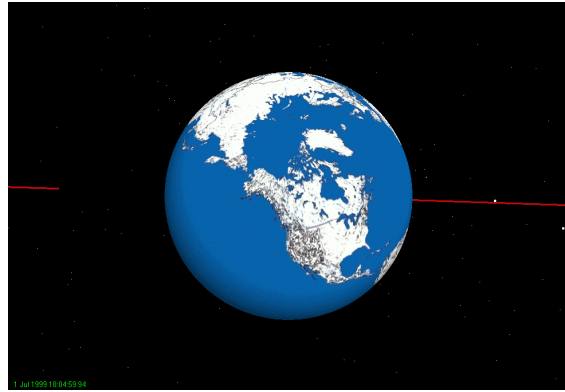
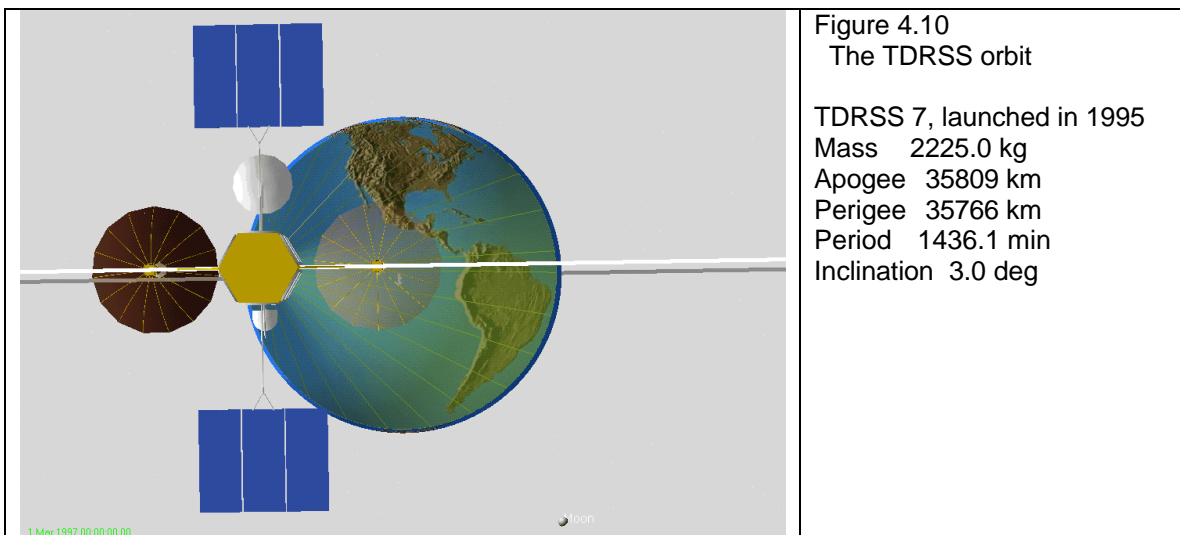
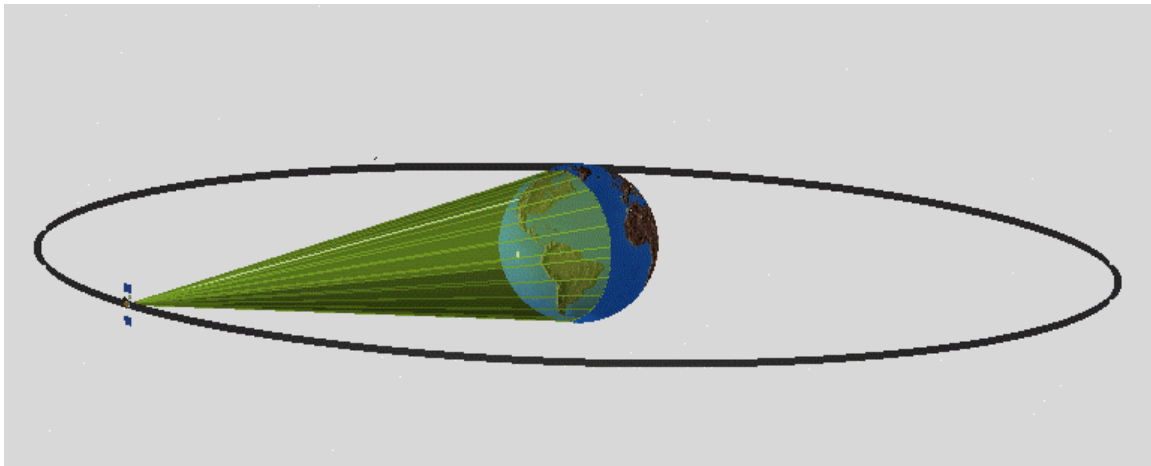


Figure 4.9b Some twelve hours later, the view from the apogee over the US – the subsolar point is in the Caribbean. (18:05 UT)

4 Geosynchronous (GEO)

Geosynchronous orbit is the standard orbit for most of the commercial and military communications satellites, the NASA telemetry system (TDRSS), and weather satellites (GOES). There is some sloppiness in the usage of the term "geosynchronous", and it is frequently interchanged with geo-stationary. Most properly, the first really means an orbit with a 24 hour period, while the latter means that the satellite position with respect to the ground is unchanging. A truly geostationary orbit is difficult to obtain, and deviations from 0° inclination of a few degrees are typical. This leads to a modest amount of north-south motion during the day.



Orbit	LEO	MEO	HEO (Molniya)	GEO
Typical Satellite	Landsat 7	GPS 2-27	Russian communications	TDRSS 7
Launch Date	04/16/1999	09/12/1996		07/13/1995
Altitude: Apogee	703 km	20314 km	39850 km	5.6 R _E 35809 km
Perigee	701 km	20047 km	500 km	35766 km
Radius: Apogee		4.15 R _E	7.2 R _E 46,228 km	6.6 R _E
Perigee			6878.1 km	
Semi-major axis	1.1 R _E	4.15 R _E 26378 km	26553.4 km	6.6 R _E
Period	98.8 minutes	12 hour 717.9 min	~12 hours 717.7 min 43,061 s	24 hours 1436.1 min 86,164 s
Inclination	98.21°	54.2°	63.4	2.97°
Eccentricity	0.00010760	0.00505460	0.741	0.000765
Mean Motion (Revs/day)	14.5711	2.00572	2.00643	1.0027
Mass	2170 kg	1881.0 kg	-	2225.0 kg

Numbers primarily from Satellite Tool Kit data base

F Problems

1. Calculate the angular velocity for a geosynchronous orbit, in radians/second.
2. Calculate the period for a circular orbit at an altitude of 1 earth radius.
3. Calculate the period for a circular orbit at the surface of the earth. What is the velocity? This is a “Herget” orbit, and is considered undesirable for a satellite.
4. Look up the orbits for the 9 planets, and plot the period vs. the semi-major axis. Do they obey Kepler’s third law? Best done by using a log-log plot. Even better, plot the two-thirds root of the period vs the semi-major axis (or mean radius).
5. Derive the radius of orbit for a geosynchronous orbit.